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The flow about the trailing edge of a supersonic oscillating aerofoil

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A description is given of the high Reynolds number ($R \gg 1$) laminar fluid motion in the neighbourhood of the trailing edge of a flat plate undergoing small amplitude sinusoidal oscillations in a uniform supersonic stream. It is shown that for oscillations of frequency $\omega^* = O(R^{\frac{1}{4}})$ and amplitude $h^* = O(R^{-\frac{1}{2}})$ a rational description of the flow at the trailing edge is based on a 'triple-deck' structure, which is a familiar feature of steady trailing-edge flows. The theory may be extended in a straightforward manner to include slow oscillations of the plate, and it is shown in general that the occurrence of separation at the trailing edge is dependent upon the magnitude of the product of the frequency and amplitude of oscillation, and that if $\omega^* \leqslant O(R^{\frac{1}{4}})$ then the flow is maintained right up to the trailing edge provided that $h^*\omega^* \ll R^{-\frac{1}{4}}$. The precise condition for the occurrence of separation is found for frequencies in the range $\omega^* \ll R^{\frac{1}{4}}$.

1. Introduction

Simultaneous investigations by Stewartson (1969) and Messiter (1970) have shown that in the limit as the Reynolds number becomes large the laminar flow at the trailing edge of a flat plate aligned with a uniform stream has a three-layered or 'triple-deck' structure which extends a distance $O(R^{-\frac{3}{8}}l)$ around the trailing edge. Here R is a representative Reynolds number for the flow based on the plate length l. Extensive work on the trailing-edge region has followed the original Stewartson–Messiter theory. The so-called 'lower-deck' equations have been solved numerically for the case of an incompressible mainstream by both Van de Vooren & Veldman (1975) and Burggraf & Jobe (1974) and for a supersonic mainstream by the present author (1974*a*). These investigations provide a quantitative evaluation of the leading-order correction to the Blasius drag on the plate. Brown & Stewartson (1970) considered the case of a flat plate at incidence and used the triple-deck theory to provide an estimate of the viscous correction to the circulation determined by the Kutta condition. It was shown that the critical angle of incidence for the occurrence of separation at the trailing edge was $O(R^{-\frac{1}{16}})$ for an incompressible mainstream and $O(R^{-\frac{1}{4}})$ for a supersonic mainstream. A quantitative evaluation of the critical angle in the latter case was made by Daniels (1974*b*, hereafter referred to as I).

Brown & Daniels (1975, hereafter referred to as II) extended the previous triple-deck investigations to consider an unsteady trailing-edge flow. A description was given of the incompressible fluid motion in the neighbourhood of the trailing edge of a flat plate performing high frequency, small amplitude oscillations in a uniform stream. Both pitching and plunging motions were considered and it was shown that for oscillations of frequency $\omega^* = O(R^{\frac{1}{4}})$ and amplitude $h^* = O(R^{-\frac{9}{16}})$ a rational description of the flow at the trailing edge is based on a complicated subdivision of the boundary layer above the plate. This consisted essentially of a triple-deck structure but also introduced the notion of a region, termed the foredeck, of height $O(R^{-\frac{1}{2}}l)$ and streamwise extent $O(R^{-\frac{1}{4}}l)$. In all, five distinct regions were distinguished at the trailing edge. In four of these, asymptotic analytic solutions were found, whilst in the fifth, the lower deck of the triple deck, an approximate linearized solution yielded an estimate for the time-dependent viscous correction to the circulation determined by the Kutta condition.

The present paper extends the work of II to the case of a supersonic mainstream and also to slow oscillations of the plate; the latter is found to be a straightforward extension of the high frequency case because of the absence of the foredeck region which occurs in the incompressible problem. Initially, however, we consider the supersonic analogue of II. A flat plate of length l is held fixed at its mid-point and performs sinusoidal oscillations of small amplitude h^* and high frequency ω^* about a line parallel to the direction of the stream at infinity. The justification for considering a flat-plate aerofoil, so that the flow remains unseparated until it enters the trailing-edge region, involves restrictions on the thickness of the aerofoil which are discussed by Brown & Stewartson (1970). The parameters of the problem are the Reynolds number $R = U_\infty l/\nu_\infty$, where U_∞ is the mainstream speed and ν_∞ is the kinematic viscosity at infinity, the non-dimensional amplitude $\tilde{h} = h^*/l$ and the frequency parameter $\tilde{S} = \omega^* l/U_\infty$. The Reynolds number is assumed to be large and the orders of magnitude of the other two parameters are chosen in terms of R. The flow is to enter a triple deck of streamwise extent $O(R^{-\frac{3}{8}}l)$ centred on the trailing edge which consists of an upper deck of height $O(R^{-\frac{3}{8}}l)$, a main deck of height $O(R^{-\frac{1}{2}}l)$ and a lower deck of height $O(R^{-\frac{5}{8}}l)$. Upstream of the triple deck the flow will be a perturbation to that of Blasius and since ω^* is large there will be a Stokes layer in the neighbourhood of the wall of thickness $O([\nu_\infty/\omega^*]^{\frac{1}{2}})$. As in II, we choose the order of magnitude of $\tilde{S}$ such that the Stokes layer and the lower deck are of the same thickness: thus $\tilde{S} = O(R^{\frac{1}{4}})$. If the order of magnitude of $\tilde{S}$ is smaller than $R^{\frac{1}{4}}$ the flow is essentially a perturbation of that for a steady aerofoil at incidence and is discussed in § 6. If it is larger, then it is probable that the triple-layered flow near the trailing edge is destroyed by the rapid oscillation. Once the order of magnitude of $\tilde{S}$ has been determined, that of $\tilde{h}$ is implied by the size of the adverse pressure gradient induced at the trailing edge by the oscillation, which should be of the same order of magnitude as the favourable pressure gradient induced by the triple deck. It emerges that for this to hold in the supersonic case we must have $\tilde{h} = O(R^{-\frac{1}{2}}l)$.

The next three sections are concerned with the three major flow regions. After the application of the method of Fourier transforms by von Kármán (1935) the development of a general linearized theory of unsteady supersonic flow advanced considerably with the work of Gunn (1947), Miles (1949) and Stewartson (1950), and the results for the inviscid flow used in § 2 have been proved using a variety of different methods. A general account of this work is given by Temple (1953). The boundary layer set up along the plate is described in § 3 and is shown to be a time-dependent perturbation of a basic Blasius flow (in suitably chosen co-ordinates), the velocity profile being accompanied by temperature and density profiles which also contain time-dependent perturbations. Final adjustment of both the velocity and temperature profiles to their specified values on the plate occurs through an inner Stokes layer.

In contrast to the incompressible flow of II, the flow just upstream of the trailing edge in the supersonic case is quite regular, with the result that the region termed the foredeck plays no role in the present study and the boundary-layer flow matches with a conventional triple-deck region at the trailing edge, which is the subject of § 4. The triple deck smoothes out the discontinuity which exists between the values of the pressure above and below the plate as the trailing edge is approached. For a frequency of oscillation ω^* of $O(R^{\frac{1}{4}})$, as in the incompressible case, this is achieved, as described above, if we assume that the amplitude of oscillation h^* is $O(R^{-\frac{1}{4}}l)$. Smaller values of h^* will merely result in a perturbation to the solution for a steady aligned plate described by Daniels (1974*a*) whilst it may be supposed that for larger values of h^* the flow will have separated before the trailing edge is reached. Certain properties of the lower-deck solution are derived in § 5. The form of the wake as it leaves the trailing-edge region may be derived from an asymptotic solution of the lower-deck equations for large values of the scaled streamwise co-ordinate. Whilst a full computational solution of the equations is not attempted, a linearized theory yields an approximate solution for the antisymmetric part of the pressure and the symmetric part of the skin friction.

In the last section we consider the modifications required to include the entire frequency range $O < \omega^* \leqslant O(R^{\frac{1}{4}})$. It is shown that the trailing-edge flow structure is basically unchanged and that if $\omega^* \ll R^{\frac{1}{4}}$ the problem effectively reduces to that of a steady plate at incidence. It may be deduced that the occurrence of separation at the trailing edge is dependent upon the magnitude of the product of the amplitude and frequency of oscillation and that if $h^*\omega^* \ll R^{-\frac{1}{4}}$ the flow is maintained right up to the trailing edge. For frequencies $\omega^* \ll R^{\frac{1}{4}}$ we may use the results for a steady plate at incidence (I) to provide a precise condition for the occurrence of separation at the trailing edge.

2. The external inviscid flow

Consider a flat plate of length l with mid-point at the origin O of a set of Cartesian co-ordinates (x^*, y^*) fixed in space. The plate, which is maintained at a constant temperature T_w, oscillates in a compressible fluid of density ρ and temperature T which has uniform velocity U_∞ and Mach number $M_\infty\,(>1)$ at infinity. At any time t^* the equation of the plate is

$$y^* = -2h^*l^{-1}x^*e^{i\omega^*t^*} \quad (-\tfrac{1}{2}l \leqslant x^* \leqslant \tfrac{1}{2}l). \tag{2.1}$$

If terms $O(h^{*2})$ are neglected then the value of the perturbation potential $\phi^*(x^*, y^*, t^*)$ on the upper surface of the plate is a special case of one of the classical results of unsteady linearized supersonic flow theory obtained by Possio (1937), Hönl (1944), Miles (1947) and others (see Temple 1953):

$$\phi^*(x^*,0+,t^*) = \frac{2}{(M_\infty^2-1)^{\frac{1}{2}}}\left(\frac{h^*}{l}\right)e^{i\omega^*t^*}\int_{x'=-\frac{1}{2}l}^{x'=x^*}(i\omega^*x'+U_\infty) \times \exp\left[\frac{i\omega^*M_\infty^2(x'-x^*)}{(M_\infty^2-1)\,U_\infty}\right]J_0\left[\frac{\omega^*M_\infty(x^*-x')}{(M_\infty^2-1)\,U_\infty}\right]dx'. \quad (2.2)$$

The pressure and slip velocity on the upper surface may then be determined by use of Bernoulli's equation as

$$p^* = p_\infty - \rho_\infty\left(\frac{\partial\phi^*}{\partial t^*}+U_\infty\frac{\partial\phi^*}{\partial x^*}\right), \quad u^* = U_\infty + \frac{\partial\phi^*}{\partial x^*}, \quad (2.3)$$

where p_∞ and ρ_∞ are the unperturbed values of the pressure and density in the external flow and (u^*, v^*) are the Cartesian velocity components.

The next four sections will be concerned with large values of the parameter ω^* and so we express (2.2) in the simplified asymptotic form

$$\phi^*(x^*,0+,t^*) = \frac{2U_\infty}{M_\infty}\left(\frac{h^*}{l}\right)x^*e^{i\omega^*t^*}\left\{1-\left[\frac{2U_\infty}{\pi M_\infty\omega^*(x^*+\frac{1}{2}l)}\right]^{\frac{1}{2}}\exp(-iM_\infty X^*) \times [M_\infty\cos(X^*-\tfrac{1}{4}\pi)+i\sin(X^*-\tfrac{1}{4}\pi)]+O\left(\frac{1}{\omega^*[x^*+\frac{1}{2}l]}\right)\right\}$$
$$(\omega^*\to\infty, \ -\tfrac{1}{2}l < x^* \leqslant \tfrac{1}{2}l), \quad (2.4)$$

where $X^* = \omega^*M_\infty(x^*+\frac{1}{2}l)/U_\infty(M_\infty^2-1)^{\frac{1}{2}}$. The expansion clearly fails at the leading edge, where $x^*+\frac{1}{2}l = O(\omega^{*-1})$, but is valid along the remainder of the upper surface of the plate. The corresponding pressure and slip velocity are now determined to first order from (2.3) as

$$\left.\begin{aligned} p^*(x^*,0+,t^*) &= p_\infty - \frac{2i\rho_\infty U_\infty\omega^*x^*}{M_\infty}\left(\frac{h^*}{l}\right)e^{i\omega^*t^*}+O(\omega^{*\frac{1}{2}}) \\ u^*(x^*,0+,t^*) &= U_\infty + \frac{2U_\infty}{M_\infty}\left(\frac{h^*}{l}\right)e^{i\omega^*t^*}+O(\omega^{*-\frac{1}{2}}) \end{aligned}\right\} (\omega^*\gg 1, -\tfrac{1}{2}l < x^* \leqslant \tfrac{1}{2}l). \quad (2.5)$$

At this stage we specify the orders of magnitude of the two parameters ω^* and h^* in terms of the Reynolds number

$$R = \epsilon^{-8} = U_\infty l/\nu_\infty. \quad (2.6)$$

Here we introduce the small parameter ϵ for convenience. First, the order of magnitude of ω^* is chosen such that the inner Stokes layer is of the same thickness as the lower deck of the triple deck at the trailing edge and thus, exactly as in the incompressible problem, we set $\omega^* = O(\epsilon^{-2})$. The critical order of magnitude of the amplitude h^* is now determined from the requirement that the pressure perturbation (2.5) should match with the order-ϵ^2 perturbation, which is a fundamental property of the triple-deck formulation, at the trailing edge. Thus $h^* = O(\epsilon^4 l)$ and the parameters ω^* and h^* are replaced by the corresponding non-dimensional scaled parameters S_0 and h_0 defined by

$$h^* = \epsilon^4 h_0 l, \quad \tilde{S} = \omega^* l/U_\infty = S_0/\epsilon^2, \quad (2.7)$$

where h_0 and S_0 are assumed to be independent of the Reynolds number. The formulae (2.5) may now be expressed in the form

$$\left.\begin{aligned}\frac{p^*(x^*,0+,t^*)-p_\infty}{\rho_\infty U_\infty^2} &= -2i\epsilon^2\frac{h_0 S_0}{M_\infty}\left(\frac{x^*}{l}\right)e^{it}\\ \frac{u^*(x^*,0+,t^*)}{U_\infty} &= 1+\epsilon^4\left[\frac{2h_0}{M_\infty}e^{it}+D_1\left(\frac{x^*}{l}\right)\right]\end{aligned}\right\}\quad (-\tfrac{1}{2}l < x^* \leqslant \tfrac{1}{2}l), \tag{2.8}$$

where $t = \omega^* t^*$, and the corresponding results for the lower surface of the plate are obtained simply by changing the sign of h_0. The steady function $D_1(x^*/l)$ must now be included since the flow due to the displacement thickness of the boundary layer on a finite plate is $O(\epsilon^4)$ (Kuo 1953).

Finally, the orders of magnitude of the total lift L and total moment M on the oscillating plate may be considered. The leading term (2.8) in the pressure gives no contribution to the overall lift on the plate, the first contribution instead arising from the second term in the asymptotic expansion (2.4), which implies that $L = O(\epsilon^3)$. However, the contribution of (2.8) to the moment about the mid-point of the plate is non-zero, so that the leading term is $O(\epsilon^2)$ and an explicit expression for M is possible:

$$\frac{M}{\rho_\infty l^2 U_\infty^2} = \epsilon^2\frac{h_0 S_0}{12M_\infty}\sin\omega^* t^* + O(\epsilon^3)\quad (\epsilon\to 0). \tag{2.9}$$

3. The perturbed boundary-layer flow

It is convenient to study the boundary-layer flow with respect to axes fixed in the plate with origin at the trailing edge and we therefore use co-ordinates $(\tilde{x},\tilde{y})$ and velocity components $(\tilde{u},\tilde{v})$ defined by

$$x^*/l-\tfrac{1}{2} = \tilde{x}+\tilde{h}\tilde{y}\,e^{it},\quad y^*/l = \tilde{y}-\tilde{h}(\tilde{x}+\tfrac{1}{2})\,e^{it}, \tag{3.1}$$

$$u^*-\tilde{h}v^*\,e^{i\omega^* t^*} = U_\infty(\tilde{u}+i\tilde{h}\tilde{S}\tilde{y}\,e^{it}),\quad \tilde{h}u^*\,e^{i\omega^* t^*}+v^* = U_\infty(\tilde{v}-i\tilde{h}\tilde{S}[\tilde{x}+\tfrac{1}{2}]\,e^{it}). \tag{3.2}$$

Although the Navier–Stokes equations are modified by additional terms to account for the rotation (see Shen & Crimi 1965) these are of sufficiently low order not to affect the solution considered here. Thus the leading terms in the boundary-layer flow set up by the external flow (2.8) satisfy the equations

$$\frac{S_0}{\epsilon^2}\frac{\partial\tilde{u}}{\partial t}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+v_1\frac{\partial\tilde{u}}{\partial y_1} = -\frac{\rho_\infty}{\rho}\frac{\partial\tilde{p}}{\partial\tilde{x}}+\frac{1}{\rho}\frac{\partial}{\partial y_1}\left(\frac{\mu\rho_\infty}{\mu_\infty}\frac{\partial\tilde{u}}{\partial y_1}\right), \tag{3.3}$$

$$\frac{S_0}{\epsilon^2}\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\tilde{x}}(\rho\tilde{u})+\frac{\partial}{\partial y_1}(\rho v_1) = 0, \tag{3.4}$$

$$\rho T = \rho_\infty T_\infty, \tag{3.5}$$

$$\frac{S_0}{\epsilon^2}\frac{\partial T}{\partial t}+\tilde{u}\frac{\partial T}{\partial\tilde{x}}+v_1\frac{\partial T}{\partial y_1}-\frac{M_\infty^2 T_\infty(\gamma-1)}{\rho}\left\{\frac{S_0}{\epsilon^2}\frac{\partial\tilde{p}}{\partial t}+\tilde{u}\frac{\partial\tilde{p}}{\partial\tilde{x}}\right\} = \frac{1}{\rho\sigma}\frac{\partial}{\partial y_1}\left(\frac{\mu\rho_\infty}{\mu_\infty}\frac{\partial T}{\partial y_1}\right)+\frac{(\gamma-1)M_\infty^2 T\mu}{\mu_\infty}\left(\frac{\partial\tilde{u}}{\partial y_1}\right)^2, \tag{3.6}$$

where
$$\tilde{p}(\tilde{x},y_1,t) = \frac{p^*-p_\infty}{\rho_\infty U_\infty^2}(x^*,0+,t^*) = -2i\epsilon^2\frac{h_0 S_0}{M_\infty}(\tilde{x}+\tfrac{1}{2})\,e^{it}, \tag{3.7}$$

μ is the coefficient of viscosity, γ is the ratio of specific heats and σ is the Prandtl number; the boundary-layer variables y_1 and v_1 are defined as

$$y_1 = \tilde{y}/\epsilon^4, \quad v_1 = \tilde{v}/\epsilon^4. \tag{3.8}$$

A full account of the derivation of these compressible boundary-layer equations may be found in Stewartson (1964, p. 15). Equation (3.5) combines the y_1-momentum equation (which infers that $\tilde{p}$ is independent of y_1 across the layer) and the equation of state. Equation (3.6) is the energy equation. The system must be completed by an equation relating μ and T and here, for simplicity, we assume that Chapman's viscosity law holds:

$$\mu/\mu_\infty = C(T/T_\infty), \tag{3.9}$$

where C is a constant determined from the 'correct' viscosity law and the conditions at the plate (see also Stewartson 1964). Here we shall assume a linear law and thus take $C = \mu_w T_\infty/\mu_\infty T_w$. This formulation provides a description of the flow which is accurate near the plate (but may be less accurate at the outer edge of the boundary layer).

The chief advantage of the assumption (3.9) is that the momentum equation (3.3) is rendered formally independent of T (or ρ) to leading order [see (3.13) below] and thus the velocity profile in the layer is determined independently of the energy equation. Once the velocity profile has been obtained, solution of (3.6) provides the temperature and density profiles across the layer. The system (3.3)–(3.9) is simplified using a generalization of the Dorodnitsyn–Howarth transformation due to Stewartson (1951) and Moore (1951), which defines the new co-ordinate $\overline{y}_1$ and the stream function $\overline{\psi}$ by the relations

$$\left.\begin{aligned} \rho &= \rho_\infty \left(\frac{\partial \overline{y}_1}{\partial y_1}\right)_{\tilde{x}}, \quad \tilde{u} = \left(\frac{\partial \overline{\psi}}{\partial \overline{y}_1}\right)_{\tilde{x},t}, \\ v_1 &= -\frac{\rho_\infty}{\rho}\left\{\left(\frac{\partial \overline{\psi}}{\partial \tilde{x}}\right)_{\overline{y}_1,t} + \tilde{u}\left(\frac{\partial \overline{y}_1}{\partial \tilde{x}}\right)_{y_1,t} + \frac{S_0}{\epsilon^2}\left(\frac{\partial \overline{y}_1}{\partial t}\right)_{\tilde{x},y_1}\right\}. \end{aligned}\right\} \tag{3.10}$$

The momentum and energy equations then become

$$\frac{S_0}{\epsilon^2}\frac{\partial^2 \overline{\psi}}{\partial \overline{y}_1 \partial t} + \frac{\partial \overline{\psi}}{\partial \overline{y}_1}\frac{\partial^2 \overline{\psi}}{\partial \tilde{x}\partial \overline{y}_1} - \frac{\partial^2 \overline{\psi}}{\partial \overline{y}_1^2}\frac{\partial \overline{\psi}}{\partial \tilde{x}} = \frac{2\epsilon^2 i \rho_\infty h_0 S_0 e^{it}}{\rho M_\infty} + C\frac{\partial^3 \overline{\psi}}{\partial \overline{y}_1^3} \tag{3.11}$$

and

$$\frac{S_0}{\epsilon^2}\frac{\partial T}{\partial t} + \frac{\partial \overline{\psi}}{\partial \overline{y}_1}\frac{\partial T}{\partial \tilde{x}} - \frac{\partial \overline{\psi}}{\partial \tilde{x}}\frac{\partial T}{\partial \overline{y}_1} - \frac{2M_\infty T(\gamma-1)h_0 S_0^2}{\rho_\infty}(\tilde{x}+\tfrac{1}{2})e^{it} = C\frac{\partial^2 T}{\partial \overline{y}_1^2} + CT_\infty M_\infty^2(\gamma-1)\left(\frac{\partial^2 \overline{\psi}}{\partial \overline{y}_1^2}\right)^2, \tag{3.12}$$

where terms $O(\epsilon^4)$ and $O(\epsilon^2)$ respectively have been neglected and the Prandtl number has been taken as unity.

The solution of (3.11) which matches with the external velocity (2.8) as $\overline{y}_1 \to \infty$ is

$$\tilde{u} = f'(\zeta) + \epsilon^4\left\{\frac{2T_0(\zeta)h_0 e^{it}}{T_\infty M_\infty} + D_0(\tilde{x}, y_1)\right\} + \ldots, \tag{3.13}$$

where $\zeta = \bar{y}_1/[2C(1+\tilde{x})]^{\frac{1}{2}}$, $f(\zeta)$ is the familiar Blasius function, satisfying

$$\left.\begin{aligned} f''' + ff'' = 0, \quad f(0) = f''(0) = 0, \quad f'(\infty) = 1, \\ 2^{-\frac{1}{2}}f''(0) = \lambda = 0{\cdot}3321, \end{aligned}\right\} \tag{3.14}$$

and the steady term $D_0(\tilde{x}, y_1)$ corresponds to the displacement term $D_1(x^*/l)$. The function $T_0(\zeta)$ is the leading term in the asymptotic expansion of the solution of (3.12) as $\epsilon \to 0$:

$$T(\tilde{x}, y_1, t) = T_0(\zeta) + \epsilon^2 T_1(\tilde{x}, \bar{y}_1, t) + \ldots. \tag{3.15}$$

Substitution of the two expansions (3.13) and (3.15) into (3.12) yields the equation for T_0 as

$$T_0'' + fT_0' + [(\gamma-1)M_\infty^2 T_\infty](f'')^2 = 0, \tag{3.16}$$

with boundary conditions

$$T_0(0) = T_w, \quad T_0(\infty) = T_\infty. \tag{3.17}$$

Equation (3.16) may be integrated directly to give the solution for T_0 (Crocco 1946):

$$T_0 = T_\infty + \tfrac{1}{2}(\gamma-1)M_\infty^2(f' - f'^2)T_\infty + (T_w - T_\infty)(1-f'). \tag{3.18}$$

The solution for T_1 is obtained from the balance between the first and fourth terms on the left-hand side of (3.12), which gives

$$T_1 = -\frac{2ih_0 S_0 M_\infty(\gamma-1)T_0(\zeta)}{\rho_\infty}(\tilde{x}+\tfrac{1}{2})e^{it}. \tag{3.19}$$

It is clear from the solutions (3.13) and (3.19) that the boundary conditions $\tilde{u}(\tilde{x}, 0, t) = 0$ and $T(\tilde{x}, 0, t) = T_w$ are not satisfied by the time-dependent terms as $y_1 \to 0$. It is therefore necessary to include within the conventional boundary layer an inner Stokes layer which reduces both the velocity and temperature profiles to their specified values at the plate. As for the incompressible problem, the appropriate order-one variable in this region is $y_2 = y_1/\epsilon$, and the asymptotic expansions for the velocity and temperature are

$$\tilde{u} = \epsilon\left\{\frac{\lambda T_\infty y_2}{[C(1+\tilde{x})]^{\frac{1}{2}}T_w}\right\} + \epsilon^4\left\{\frac{-\lambda^2 y_2^4}{12C^2(1+\tilde{x})^2}\left(\frac{T_w}{T_\infty}\right)^4 + u_1(\tilde{x}, y_2)e^{it}\right\} + \ldots, \tag{3.20}$$

$$\begin{aligned} T = T_w + \epsilon\left\{\frac{\lambda y_2}{[C(1+\tilde{x})]^{\frac{1}{2}}}\left(\left[1+\frac{\gamma-1}{2}M_\infty^2\right]\frac{T_\infty^2}{T_w} - T_\infty\right)\right\} \\ + \epsilon^2\left\{-\frac{y_2^2(\gamma-1)}{C(1+\tilde{x})}\left(\frac{T_\infty M_\infty \lambda}{T_w}\right)^2 + T_2(\tilde{x}, y_2)e^{it}\right\} + \ldots. \end{aligned} \tag{3.21}$$

The equation for the unknown function u_1 is obtained by substitution into (3.3) as

$$C\left(\frac{T_w}{T_\infty}\right)^2\frac{\partial^2 u_1}{\partial y_2^2} - iS_0 u_1 = -\frac{2iT_w h_0 S_0}{M_\infty T_\infty}. \tag{3.22}$$

The boundary conditions for u_1 are that $u_1(\tilde{x}, 0) = 0$ and, in order to match with the outer solution (3.13),

$$u_1 \to 2T_w h_0/T_\infty M_\infty \quad (y_2 \to \infty). \tag{3.23}$$

The appropriate solution is thus

$$u_1(\tilde{x}, y_2) = \frac{2T_w h_0}{M_\infty T_\infty}\left(1 - \exp\left\{-\frac{i^{\frac{1}{2}} S_0^{\frac{1}{2}} T_\infty}{C^{\frac{1}{2}} T_w} y_2\right\}\right), \tag{3.24}$$

where $i^{\frac{1}{2}} = (1+i)/\sqrt{2}$. The function T_2 satisfies a similar second-order equation whose solution which matches with the inner limit of (3.15) as $y_1 \to 0$ and satisfies the boundary condition on the plate, $T_2(\tilde{x}, 0) = 0$, is

$$T_2(\tilde{x}, y_2) = -\frac{2ih_0 S_0 M_\infty T_w(\gamma - 1)}{\rho_\infty}(\tilde{x} + \tfrac{1}{2})\left(1 - \exp\left\{-\frac{i^{\frac{1}{2}} S_0^{\frac{1}{2}} T_\infty}{C^{\frac{1}{2}} T_w} y_2\right\}\right). \tag{3.25}$$

4. The triple deck

The results of the previous section clearly indicate that the boundary layer on the plate approaches the trailing edge ($\tilde{x} = 0-$) in a perfectly regular manner. In incompressible flow, the foredeck results from the critical balance between the sizes of the operators $\partial/\partial t^*$ and $\partial/\partial x^*$ at the trailing edge, but in the present problem we always have $\partial/\partial t^* \gg \partial/\partial x^*$ outside the triple deck as $x^* - \frac{1}{2}l \to 0-$. The result is that the foredeck region of the incompressible problem has no counterpart in supersonic flow and the boundary-layer flow of § 3 matches with a conventional triple deck at the trailing edge in which the order-one streamwise variable is $x_2 = \tilde{x}/\epsilon^3$. The leading-order pressure variation in the triple deck is forced at $x_2 = -\infty$ by the time-dependent perturbation (2.8); however the time-dependent parts of the velocity and temperature in the boundary-layer flow match with lower-order terms in the triple-deck expansions and have no effect upon the fundamental problem at the trailing edge.

Despite the time dependence, the analytic structure of the triple deck is basically that of Stewartson & Williams (1969). In the main deck, where $y_1 = O(1)$, we write

$$\tilde{p} = \epsilon^2 p_m(x_2, t) + \epsilon^3 p_3(x_2, y_1, t) + \ldots, \tag{4.1}$$

$$\tilde{u} = U_B(y_1) + \epsilon u_m(x_2, y_1, t) + \ldots, \tag{4.2}$$

$$\tilde{v} = \epsilon^2 v_m(x_2, y_1, t) + \ldots, \tag{4.3}$$

$$\rho = R_B(y_1) + \epsilon \rho_m(x_2, y_1, t) + \ldots, \tag{4.4}$$

where $U_B(y_1) = f'(\overline{y}_1/(2C)^{\frac{1}{2}})$ and $R_B(y_1) = \rho_\infty T_\infty [T_0(\overline{y}_1/(2C)^{\frac{1}{2}})]^{-1}$ are respectively the generalized Blasius velocity profile and the density profile evaluated at the trailing edge. Since time derivatives do not enter the equations until terms of order lower than those shown explicitly in (4.1)–(4.4) are considered, the results of Stewartson & Williams are immediately applicable and we have

$$u_m = -A_m(x_2, t)\, dU_B/dy_1, \quad v_m = U_B(y_1)\,[\partial A_m(x_2, t)]/\partial x_2, \quad \rho_m = A_m(x_2, t)\, dR_B/dy_1, \tag{4.5}$$

where the match with the boundary-layer flow upstream given by (3.13), (3.15) and (2.8) implies that

$$A_m \to 0, \quad p_m \to -(ih_0 S_0/M_\infty)\, e^{it} \quad (x_2 \to -\infty). \tag{4.6}$$

A further result we shall require is that

$$p_3 - (\partial^2 A_m/\partial x_2^2)\, y_1 = O(1) \quad (y_1 \to \infty). \tag{4.7}$$

In the upper deck, where $y_0 = \tilde{y}/\epsilon^3 = O(1)$, we write

$$\tilde{p} = \epsilon^2 p_u + \ldots, \quad \tilde{u} = 1 + \epsilon^2 u_u + \ldots, \quad \tilde{v} = \epsilon^2 v_u + \ldots. \tag{4.8}$$

The function p_u then satisfies the Prandtl–Glauert equation in the variables x_2 and y_0, with solution

$$p_u(x_2, y_0, t) = F_u([x_2 - y_0(M_\infty^2 - 1)^{\frac{1}{2}}], t). \tag{4.9}$$

From (4.1) and (4.7), the match with the main deck as $y_0 \to 0$ and $y_1 \to \infty$ then provides the relations

$$F_u(x_2, t) = p_m(x_2, t), \quad -(M_\infty^2 - 1)^{\frac{1}{2}}\,\partial F_u/\partial x_2 = -\partial^2 A_m/\partial x_2^2, \tag{4.10}$$

which combine after one integration to provide the basic relation between p_m and A_m:

$$p_m(x_2, t) = \frac{1}{(M_\infty^2 - 1)^{\frac{1}{2}}}\frac{\partial A_m}{\partial x_2} - \frac{ih_0 S_0}{M_\infty} e^{it}, \tag{4.11}$$

where the time-dependent constant of integration follows from the upstream boundary conditions (4.6).

In the lower deck, $y_2 = y_1/\epsilon = O(1)$ and we have

$$\tilde{p} = \epsilon^2 p_l(x_2, t) + \ldots, \tag{4.12}$$

$$\tilde{u} = \epsilon u_l(x_2, y_2, t) + \ldots, \tag{4.13}$$

$$\tilde{v} = \epsilon^3 v_l(x_2, y_2, t) + \ldots, \tag{4.14}$$

$$T = T_w + \epsilon T_l(x_2, y_2, t) + \ldots, \quad \rho = \rho_w + \epsilon \rho_l(x_2, y_2, t) + \ldots. \tag{4.15}$$

It may be noted from (4.15) that the leading terms in the temperature and density expansions are in fact equal to their respective values at the plate throughout the lower deck, despite the fact that the plate terminates at $x_2 = 0$. This is because the leading terms do not violate the downstream condition of continuous flow across the wake. The match with the main deck of the triple deck now provides the relations

$$\left.\begin{aligned} p_l(x_2, t) &= p_m(x_2, t) \\ u_l &\sim \lambda(T_\infty/T_w) C^{-\frac{1}{2}} y_2 - \lambda A_m(x_2, t)\ (T_\infty/T_w)\, C^{-\frac{1}{2}} \end{aligned}\right\} \quad (y_2 \to \infty), \tag{4.16}$$

with the result that the fundamental problem for the lower deck for a supersonic high frequency oscillating plate is to solve the equations

$$S\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.17}$$

where

$$p(x, t) = \begin{cases} p_T(x, t) = \partial A_T/\partial x - ihS\, e^{it}, & (y > 0), \quad (4.18) \\ p_B(x, t) = -\partial A_B/\partial x + ihS\, e^{it}, & (y < 0), \quad (4.19) \end{cases}$$

subject to the boundary conditions

$$u \to |y|, \quad p_T \to -ihS\, e^{it}, \quad p_B \to ihS\, e^{it} \quad (x \to -\infty), \tag{4.20}$$

$$u = v = 0 \quad \text{on} \quad y = 0 \quad (x < 0), \tag{4.21}$$

$$u - y \to -A_T(x, t) \quad (y \to +\infty), \tag{4.22}$$

$$u + y \to +A_B(x, t) \quad (y \to -\infty), \tag{4.23}$$

$$p_T = p_B, \quad u, v \text{ smooth for all } y \quad (x > 0), \tag{4.24}$$

$$p \to 0 \quad (x \to \infty). \tag{4.25}$$

The final condition (4.25) is needed to ensure a match with the solution in the wake, as for the steady supersonic problems of Daniels (1974*a*, *b*). The parameters λ, C, M_∞ and T_w/T_∞ have been eliminated by means of the transformations

$$\left.\begin{aligned} p_m &= C^{\frac{1}{4}}\lambda^{\frac{1}{2}}(M_\infty^2-1)^{-\frac{1}{4}}p, & u_l &= C^{\frac{1}{8}}\lambda^{\frac{1}{4}}(M_\infty^2-1)^{-\frac{1}{8}}(T_w/T_\infty)^{\frac{1}{2}}u,\\ v_l &= C^{\frac{3}{8}}\lambda^{\frac{3}{4}}(M_\infty^2-1)^{\frac{1}{8}}(T_w/T_\infty)^{\frac{1}{2}}v, & x_2 &= C^{\frac{3}{8}}\lambda^{-\frac{5}{4}}(M_\infty^2-1)^{-\frac{3}{8}}(T_w/T_\infty)^{\frac{3}{2}}x,\\ y_2 &= C^{\frac{5}{8}}\lambda^{-\frac{3}{4}}(M_\infty^2-1)^{-\frac{1}{8}}(T_w/T_\infty)^{\frac{3}{2}}y, & A_m &= C^{\frac{5}{8}}\lambda^{-\frac{3}{4}}(M_\infty^2-1)^{-\frac{1}{8}}(T_w/T_\infty)^{\frac{3}{2}}A,\\ S_0 &= C^{-\frac{1}{4}}\lambda^{\frac{3}{2}}(T_w/T_\infty)^{-1}(M_\infty^2-1)^{\frac{1}{4}}S, & h_0 &= M_\infty\lambda^{-1}(T_w/T_\infty)(M_\infty^2-1)^{-\frac{1}{2}}h. \end{aligned}\right\} \quad (4.26)$$

At this point it is noted that, as for the incompressible problem, the trailing-edge theory for a plate oscillating in the plunging mode follows immediately from that for the pitching mode with an appropriate relation between the amplitudes H^* and h^* (respectively). For the supersonic plunging mode we write the equation of the plate as

$$y^* = -H^* e^{i\omega^* t^*} \quad (-\tfrac{1}{2}l \leqslant x^* \leqslant \tfrac{1}{2}l). \tag{4.27}$$

It then follows that the pressure on the upper surface is given by

$$\frac{p^*-p_\infty}{\rho_\infty U_\infty^2}(x^*, 0+, t^*) = -i\,\epsilon^2 \frac{H_0 S_0}{M_\infty} e^{it}, \tag{4.28}$$

where $H^*/l = \tilde{H} = \epsilon^4 H_0$ and S_0 is defined by (2.7); transformation to co-ordinates on the plate is achieved by use of equations (8.5) and (8.6) of II with $\frac{1}{2}\tilde{H}$ replaced by $\tilde{H}$. Comparison of the above result with the corresponding formula (2.8) evaluated at $x^* = \frac{1}{2}l$ (the trailing edge) shows that all the results of this section apply to the plunging mode (4.27) if h_0 is replaced by H_0. Thus, in contrast to the incompressible result, the trailing-edge effect of a supersonic pitching motion is equivalent to that of a plunging motion of *equal* (instead of halved) amplitude. As far as overall comparisons are concerned, an increase in lift and decrease in moment for the plunging motion similar to those in incompressible flow are found, for it follows from (4.28) that $M = O(\epsilon^3)$ whilst the total lift L on the plate is given by

$$\frac{L}{\rho_\infty l U_\infty^2} = \frac{2\epsilon^2 H_0 S_0}{M_\infty}\sin\omega^* t^* + O(\epsilon^3) \quad (\epsilon \to 0). \tag{4.29}$$

5. The lower-deck solution

We now consider the solution of the fundamental problem (4.17)–(4.25). Asymptotically, the most interesting part of the flow is where $x \to \infty$ for this shows the form assumed by the wake vortex sheet as it leaves the trailing-edge region. A simple transformation is sufficient to show that, as in the incompressible problem, the lower deck merges into the inner form of Goldstein's (1930) wake solution, which has an appropriate time-dependent displacement of the centre-line. We replace the variables y and v by new variables $\bar{y}$ and $\bar{v}$ defined by

$$\bar{y} = y - \theta(x,t), \quad \bar{v} = v - S\,\partial\theta/\partial t - u\,\partial\theta/\partial x. \tag{5.1 a, b}$$

Choice of the function $\theta(x,t)$ as

$$\theta(x,t) = ihSx\,e^{it} \tag{5.2}$$

now reduces the conditions (4.18), (4.19), (4.22) and (4.23) to the single equation

$$u-|\bar{y}| \to -\int_0^x p(t)\,dt - \tfrac{1}{2}[A_T(0,t)-A_B(0,t)] - \tfrac{1}{2}[A_T(0,t)+A_B(0,t)]\operatorname{sgn}\bar{y} \quad (|\bar{y}|\to\infty,\ x>0), \tag{5.3}$$

whilst substitution of (5.1) into (4.17) leaves both equations unchanged, except that y and v are replaced by $\bar{y}$ and $\bar{v}$ respectively. The leading term of the asymptotic solution of these equations as $x\to\infty$ which satisfies the condition (5.3) is

$$u = x^{\frac{1}{3}}\bar{f}_0'(\bar{\eta})+\ldots, \quad p = -0.297x^{-\frac{2}{3}}+\ldots, \tag{5.4}$$

where $\bar{\eta} = \bar{y}/x^{\frac{1}{3}}$. The function $\bar{f}_0$ is identical with the leading term of the inner solution of the Goldstein wake and satisfies

$$\left.\begin{array}{c}\bar{f}_0''' + \tfrac{2}{3}\bar{f}_0\bar{f}_0'' - \tfrac{1}{3}\bar{f}_0'^2 = 0, \quad \bar{f}_0(0) = \bar{f}_0''(0) = 0, \quad \bar{f}_0'(0) = 1{\cdot}611. \\ \bar{f}_0' \sim \bar{\eta} \quad (\bar{\eta}\to\infty).\end{array}\right\} \tag{5.5}$$

The asymptotic solution of the energy equation in the lower deck for T_l yields the corresponding result

$$T = T_w + \epsilon\{C^{\frac{1}{4}}\lambda^{\frac{1}{2}}(M_\infty^2-1)^{-\frac{1}{8}}(T_w/T_\infty)^{\frac{1}{2}}x^{\frac{1}{3}}\bar{f}_0'(\bar{\eta})+\ldots\} \quad (x\to\infty). \tag{5.6}$$

As far as the properties of the wake are concerned, (5.1a) indicates that it emerges from the triple-deck region where $\tilde{x} = O(\epsilon^3)$ with a linear displacement given by

$$\tilde{y} = \epsilon^5\left(\frac{\tilde{x}}{\epsilon^3}\right) iC^{\frac{1}{2}}\frac{(M_\infty^2-1)^{\frac{1}{2}}}{M_\infty}h_0 S_0 e^{it} \quad \left(\frac{\tilde{x}}{\epsilon_3}\to\infty\right), \tag{5.7}$$

and thus has a phase lag of $\frac{1}{2}\pi$ relative to the oscillation of the plate. Further, (5.1b) indicates that the wake as a whole (which has a thickness of order ϵ^4) oscillates from side to side with velocity

$$-\epsilon^3 U_\infty h_0 S_0^2 C^{\frac{1}{2}}\frac{(M_\infty^2-1)^{\frac{1}{2}}}{M_\infty}\left(\frac{\tilde{x}}{\epsilon^3}\right)e^{it} \quad \left(\frac{\tilde{x}}{\epsilon^3}\to\infty\right) \tag{5.8}$$

in this region.

We now consider the possibility of a full computational solution of the problem (4.17)–(4.25), based on the methods used for the steady problem of I. First, it is noted that the solution must display a periodic symmetry of the form

$$\left.\begin{array}{c}u(x,y,t) = u(x,-y,t+\pi), \quad p_T(x,t) = p_B(x,t+\pi), \\ v(x,y,t) = -v(x,-y,t+\pi),\end{array}\right\} \tag{5.9}$$

so that the domain of the problem may be be reduced to $y \geqslant 0$ only by replacing conditions (4.19), (4.23) and (4.24) by

$$p(x,t) = p(x,t+\pi), \quad \left.\begin{array}{c}u\\ v\\ \dfrac{\partial u}{\partial y}\end{array}\right\}(x,0,t) = \left.\begin{array}{c}\mu\\ -v\\ -\dfrac{\partial u}{\partial y}\end{array}\right\}(x,0,t+\pi) \quad (x>0) \tag{5.10}$$

and (4.20) by

$$u\to y, \quad p\to -ihS\,e^{it} \quad (x\to-\infty); \tag{5.11}$$

the terms p_B and A_B are no longer required. To reduce the problem still further, the time dependence may be eliminated completely if we represent the *real* solutions for the pressure p and stream function ψ by the Fourier series

$$\left.\begin{aligned} \psi(x,y,t) &= \tfrac{1}{2}a_0(x,y) + \sum_{n=1}^{\infty} (a_n(x,y)\cos nt + b_n(x,y)\sin nt) \quad (y \geqslant 0), \\ p(x,t) &= \tfrac{1}{2}c_0(x) + \sum_{n=1}^{\infty} (c_n(x)\cos nt + d_n(x)\sin nt), \end{aligned}\right\} \tag{5.12}$$

where the coefficients a_n, b_n, c_n and d_n are real. The initial conditions (5.11), for instance, then become

$$\left.\begin{aligned} &\tfrac{1}{2}\,\partial a_0/\partial y \to y; \quad \partial a_n/\partial y, \partial b_n/\partial y \to 0 \quad (n = 1,\ldots) \\ &c_0, c_1 \to 0, \quad d_1 \to hS; \quad c_n, d_n \to 0 \quad (n = 2,\ldots) \end{aligned}\right\} (x \to -\infty), \tag{5.13}$$

and we have a downstream marching problem of the type described in I for each of the functions a_n $(n = 0,1,\ldots)$ and b_n $(n = 1,2,\ldots)$ and the corresponding pressure coefficients c_n and d_n.

Substitution of (5.12) into (4.17) shows that the equations for the a_n and b_n are coupled and so would require a simultaneous solution at each downstream step; and the initial values of each c_n and d_n would in theory have to be adjusted using a shooting technique to satisfy the downstream condition (4.25). This procedure might be tractable for a very high truncation of the Fourier series (5.12), and certainly for small values of h a linearized solution could be computed by following a procedure outlined by the present author (1974*c*). However, we can find an analytic solution for the antisymmetric part of the pressure and the symmetric part of the skin friction using an approximate linearized theory based on the same assumptions as those of II for the incompressible case. Thus we introduce the complex perturbation quantities $W_T - W_B$, $V_T + V_B$ and $P_T - P_B$, where

$$\left.\begin{aligned} w_T - w_B &= (W_T - W_B)\,e^{it}, \quad u = u_T = Y + w_T \quad (y > 0), \\ u &= u_B = Y + w_B \quad (y < 0), \quad Y = |y| \quad (x \leqslant 0), \\ p_T - p_B &= (P_T - P_B)\,e^{it}, \quad v_T + v_B = (V_T + V_B)\,e^{it}, \end{aligned}\right\} \tag{5.14}$$

on the assumption that for small values of h we have $w_T, w_B \ll 1$ and the uniform shear represents a good first approximation to the flow upstream of the trailing edge. Substitution of the forms (5.14) into the lower-deck equations (4.17) and neglect of terms nonlinear in the perturbation quantities then result in the equations

$$\left.\begin{aligned} Y\frac{\partial}{\partial x}(W_T - W_B) + (V_T + V_B) &= -\frac{d}{dx}(P_T - P_B) + \frac{\partial^2}{\partial Y^2}(W_T - W_B), \\ \frac{\partial}{\partial x}(W_T - W_B) + \frac{\partial}{\partial Y}(V_T + V_B) &= 0. \end{aligned}\right\} \tag{5.15}$$

We now seek a solution of the form

$$\left.\begin{aligned} \tfrac{1}{2}(P_T - P_B) &= -ihS + ihS\,e^{\kappa x}, \\ \tfrac{1}{2}(W_T - W_B) = -ihS\,e^{\kappa x}F'(Y), &\quad \tfrac{1}{2}(V_T + V_B) = ihS\kappa\,e^{\kappa x}F(Y), \end{aligned}\right\} (x < 0), \tag{5.16}$$

where κ is an undetermined complex constant with $\operatorname{Re}\kappa > 0$. The form of the antisymmetric part of the pressure is chosen to satisfy the upstream boundary condition $\frac{1}{2}(P_T - P_B) \to -ihS$ $(x \to -\infty)$ and the trailing-edge condition

$$P_T(0) = P_B(0).$$

The equation for F is

$$-F'''' - \kappa = -(iS + Y\kappa)F' + \kappa F, \tag{5.17}$$

with boundary conditions

$$F(0) = F'(0) = 0, \quad F'(\infty) = 1/\kappa. \tag{5.18}$$

The first two boundary conditions and (5.17) are satisfied if

$$F'(Y) = -\frac{\kappa^{\frac{2}{3}}}{\operatorname{Ai}'\left(\dfrac{iS}{\kappa^{\frac{2}{3}}}\right)} \int_{Y'=0}^{Y'=Y} \operatorname{Ai}\left(\frac{iS + \kappa Y'}{\kappa^{\frac{2}{3}}}\right) dY', \tag{5.19}$$

where the value of the cube root is specified if we assume that $-\frac{1}{2}\pi < \arg\kappa < \frac{1}{2}\pi$. To apply the third boundary condition the further approximation that $S \gg 1$ is made, as in the incompressible case. We may then replace the Airy functions by their asymptotic forms and perform the integration of (5.19) to obtain the simple expression

$$\kappa^2 = iS \tag{5.20}$$

for the constant κ. The real solutions for the antisymmetric part of the pressure and the symmetric part of the skin friction then become

$$\left.\begin{aligned} \tfrac{1}{2}(p_T - p_B) &= -hS[e^X \sin(X+t) + \sin t], \\ \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)_T + \left(\frac{\partial u}{\partial y}\right)_B\right]_{y=0} &= hS\, e^X \sin(X+t), \end{aligned}\right\} \tag{5.21}$$

where $X = (\frac{1}{2}S)^{\frac{1}{2}}x$. Alternatively, in view of the property (5.9), the pressures on either side of the plate may be written as

$$\left.\begin{aligned} p_T(x,t) &= p_s(x) - hS[e^X \sin(X+t) + \sin t], \\ p_B(x,t) &= p_s(x) + hS[e^X \sin(X+t) + \sin t], \end{aligned}\right\} \tag{5.22}$$

where now the unknown steady symmetric part $p_s(x)$, which represents a correction to the basic linear shear solution for $h = 0$ and which is assumed small for the purposes of the linearization (5.14), must be included.

6. Low frequency oscillation and separation at the trailing edge

The solution of the preceding sections has dealt with the specific case for which the frequency of oscillation ω^* is $O(\epsilon^{-2})$ and the amplitude of oscillation h^* is $O(\epsilon^4)$. This choice of the orders of magnitude of the two parameters of the problem was shown in § 2 to produce a pressure perturbation in the boundary layer which exactly matches the perturbation which occurs in the triple-deck formulation at the trailing edge. Moreover, the leading-order perturbations to the velocity components in the triple deck decay exponentially as $x_2 \to -\infty$ and the velocity perturbations in the boundary layer, which merely force lower-order terms in the triple-deck expansions, do not affect the fundamental problem for the lower deck formulated in § 4. This property permits a straightforward extension of the theory for $\omega^* \sim \epsilon^{-2}$ to cover the trailing-edge flow for the complete range $0 \leqslant \omega^* \leqslant O(\epsilon^{-2})$, without the necessity of providing a detailed solution for the perturbed boundary-layer flow on the plate.

From the above discussion it is clear that the dominant factor which determines the trailing-edge flow is the size of the inviscid pressure on the surface of the plate as $x^* - \frac{1}{2}l \to 0-$. This may be determined for any ω^* from the formulae (2.2) and (2.3). If ω^* is large we may use the simplified form (2.5),

$$\frac{p^*-p_\infty}{\rho_\infty U_\infty^2}(\tfrac{1}{2}l, 0+, t^*) = -\frac{i}{M_\infty}\left(\frac{h^*}{l}\right)\left(\frac{\omega^* l}{U_\infty}\right)e^{i\omega^* t^*}, \tag{6.1}$$

whilst if ω^* is of order one we must use the complete formula (2.2), which gives

$$\frac{p^*-p_\infty}{\rho_\infty U_\infty^2}(\tfrac{1}{2}l, 0+, t^*) = \frac{1}{(M_\infty^2-1)^{\frac{1}{2}}}\left(\frac{h^*}{l}\right)P\left(M_\infty, \frac{\omega^* l}{U_\infty}\right)e^{i\omega^* t^*}, \tag{6.2}$$

where

$$\left.\begin{aligned} P(a,b) &= \frac{i}{b}Q(a,b,\tfrac{1}{2}) + \frac{\partial Q}{\partial c}(a,b,\tfrac{1}{2}), \\ Q(a,b,c) &= 2\int_{c'=-\frac{1}{2}}^{c'=c}(1+ibc')\exp\left(\frac{ia^2b[c'-c]}{a^2-1}\right)J_0\left(\frac{ab}{a^2-1}[c-c']\right)dc'. \end{aligned}\right\} \tag{6.3}$$

However, whichever form is used, the pressure just upstream of the trailing edge is always $O(h^*\omega^*)$ and certainly for the entire range $0 \leqslant \omega^* \leqslant O(\epsilon^{-2})$ the flow at the trailing edge is described by a triple-deck structure provided that $h^*\omega^* = O(\epsilon^2)$. This last condition should not be regarded as a restriction, but as an indication of the critical range of the parameters: if $h^*\omega^* < O(\epsilon^2)$ then the leading-order problem at the trailing edge is simply that for a steady aligned plate described in Daniels (1974*a*), whilst if $h^*\omega^* > O(\epsilon^2)$ then it may be assumed that the flow on the plate separates before the trailing edge is reached.

To fix the ideas suggested in the preceding paragraph we now consider in a little more detail the problem of a *slowly* oscillating plate, for which $\omega^* \sim 1$ and $h^* \sim \epsilon^2$, and show how it may be placed in context with both the steady inclined configuration of I and the high frequency problem considered in §§ 2–5. The external inviscid flow on the upper surface of the plate is determined from (2.2) and (2.3) and has the form

$$\left.\begin{aligned} \frac{p^*-p_\infty}{\rho_\infty U_\infty^2}(x^*, 0+, t^*) &= \frac{\epsilon^2 h_0 e^{it}}{(M_\infty^2-1)^{\frac{1}{2}}}P_e\left(M_\infty, \tilde{S}, \frac{x^*}{l}\right), \\ \frac{u^*}{U_\infty}(x^*, 0+, t^*) &= 1 + \frac{\epsilon^2 h_0 e^{it}}{(M_\infty^2-1)^{\frac{1}{2}}}U_e\left(M_\infty, \tilde{S}, \frac{x^*}{l}\right), \end{aligned}\right\} \tag{6.4}$$

where we now replace (2.7) by the scalings

$$h^* = \epsilon^2 h_0 l, \quad \tilde{S} = \omega^* l/U_\infty = S_0. \tag{6.5}$$

The complicated functions P_e and U_e may be written explicitly in an integral form similar to (2.2) but their precise behaviour will be of no concern. The major requirement from P_e will be its limiting form as $x^* - \frac{1}{2}l \to 0-$, and this is precisely the function $P(M_\infty, \tilde{S})$ given by (6.3).

The external flow (6.4) produces a boundary-layer flow on the plate where $y_1 = O(1)$ which is that of Blasius together with an order-ϵ^2 time-dependent perturbation. The perturbation to the temperature profile is also $O(\epsilon^2)$. The time-dependent parts vary across the entire layer since for $\omega^* = O(1)$ the inner layer described in § 3 effectively becomes of the same thickness as the boundary layer. We do not attempt a solution for the perturbation in the boundary layer, but note that, since the function U_e is quite regular as $x^* - \frac{1}{2}l \to 0-$, the correction to the Blasius profile just upstream of the triple deck at the trailing edge is $O(\epsilon^2)$ and thus does not affect the leading-order velocity terms in the lower and main decks.

In the upper deck we have $u_u \to h_0 U_e(M_\infty, \tilde{S}, \frac{1}{2})\, e^{it}$ as $x_2 \to -\infty$, but the solution is still given by (4.9) and the fundamental problem in the lower deck may be stated as follows: solve

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6.6}$$

where

$$p(x,t) = \begin{cases} p_T(x,t) = \partial A_T/\partial x + hP(M_\infty, \tilde{S})\, e^{it} & (y > 0) & (6.7) \\ p_B(x,t) = -\partial A_B/\partial x - hP(M_\infty, \tilde{S})\, e^{it} & (y < 0) & (6.8) \end{cases}$$

and

$$u \to |y|, \quad p_T \to hP(M_\infty, \tilde{S})\, e^{it}, \quad p_B \to -hP(M_\infty, \tilde{S})\, e^{it} \quad (x \to -\infty), \tag{6.9}$$

$$u = v = 0 \quad \text{on} \quad y = 0 \quad (x < 0), \tag{6.10}$$

$$u - y \to -A_T(x,t) \quad (y \to \infty), \qquad u + y \to A_B(x,t) \quad (y \to -\infty), \tag{6.11}$$

$$p_T = p_B, \quad u, v \text{ smooth for all } y \quad (x > 0), \quad p \to 0 \quad (x \to \infty). \tag{6.12}$$

Here the quantities x, y, u, v, p and A are defined by the transformations (4.26) and in addition we write

$$h_0 = C^{\frac{1}{4}} \lambda^{\frac{1}{2}} (M_\infty^2 - 1)^{\frac{1}{4}}\, h. \tag{6.13}$$

A comparison of the above system with the lower-deck problem (2.13)–(2.20) of I now shows that, at any *given* time t, the solution for a slowly oscillating plate is given by that for a steady inclined plate at an angle of incidence α^* if we define the scaled angle of incidence α by the relation

$$\alpha = \mathrm{Re}\{hP(M_\infty, \tilde{S})\, e^{it}\}. \tag{6.14}$$

We may now deduce the condition for the occurrence of separation at the trailing edge of the slowly oscillating plate, for the results of I indicate that, as the value of α is increased, separation first occurs when $\alpha = \alpha_c = 2{\cdot}0502$. Separation therefore occurs at the trailing edge of the slowly oscillating plate if

$$h \max_{0 \leqslant t \leqslant 2\pi} |\mathrm{Re}\{P(M_\infty, \tilde{S})\, e^{it}\}| \geqslant \alpha_c, \tag{6.15}$$

a complicated condition which depends on the non-dimensional amplitude of oscillation h^*/l, the frequency parameter $\tilde{S}$, the Mach number M_∞ and the Reynolds number R. The simplest effect to consider is that of the amplitude: for a given Reynolds number R, Mach number M_∞ and frequency parameter $\tilde{S}$, separation first occurs at the trailing edge as the amplitude is increased when $\omega^* t^* = t = t_{\max}$ [any value of t which produces the maximum of (6.14)] and

$$h^* = \frac{lC^{\frac{1}{4}}\lambda^{\frac{1}{2}}(M_\infty^2-1)^{\frac{1}{4}}}{R^{\frac{1}{4}}} 2{\cdot}050 \left\{ \max_{0 \leqslant t^* \leqslant 2\pi/\omega^*} |\mathrm{Re}\,[P(M_\infty, \tilde{S})\, e^{i\omega^* t^*}]| \right\}^{-1}, \quad (6.16)$$

where the function P is given by (6.3).

For both large and small values of the frequency ω^*, the separation law (6.15) takes a particularly simple form. As $\tilde{S} \to \infty$ we have $P \sim -i\tilde{S}(M_\infty^2-1)^{\frac{1}{2}}/M_\infty$, so that separation occurs, initially, at the times $t^* = \pi/2\omega^*$ and $3\pi/2\omega^*$ (as the plate passes through its *mean* position) if

$$h^* \geqslant \frac{2{\cdot}050 lC^{\frac{1}{4}}\lambda^{\frac{1}{2}}M_\infty}{R^{\frac{1}{4}}\tilde{S}(M_\infty^2-1)^{\frac{1}{4}}}. \quad (6.17)$$

The effect of an increase in frequency is thus to reduce the minimum amplitude at which separation occurs at the trailing edge. As $\tilde{S} \to 0$ we have $P \to 2$ and (6.15) implies that separation occurs at $t^* = 0$ if

$$\frac{2h^*}{l} \geqslant \alpha_c \frac{C^{\frac{1}{4}}\lambda^{\frac{1}{2}}(M_\infty^2-1)^{\frac{1}{4}}}{R^{\frac{1}{4}}}. \quad (6.18)$$

This merely restates the case of a steady inclined plate since the angle of incidence of an oscillating plate in its extreme position is $2h^*/l$ and the right-hand side of (6.18) is precisely the trailing-edge stall angle of I.

The separation law (6.15) holds for all h^* and ω^* provided that $h^* = O(\epsilon^2)$ and $\omega^* = O(1)$, and we have seen that in the limit as $\omega^* \to 0$ we recover the case of a steady inclined plate. We now consider the opposite limit, in which $\omega^* \to \infty$ and the oscillation of the plate becomes more rapid. Consider any ω^* in the range $1 \ll \omega^* \ll \epsilon^{-2}$. The time-dependent perturbations to the external inviscid pressure and slip velocity on the plate are $O(h^*\omega^*)$ and $O(h^*)$ respectively and produce perturbations of the same magnitude in the boundary layer and an inner Stokes layer of thickness $O(\epsilon^4\omega^{*-\frac{1}{2}})$; the latter is required to reduce the fluid velocity to zero at the plate. To obtain the critical problem in which separation may occur at the trailing edge, the order of magnitude of h^* must be such that the inviscid pressure just upstream of the trailing edge, as given by (6.1), forces the leading-order pressure perturbation in the triple deck. Thus we assume $h^* = O(\epsilon^2\omega^{*-1})$. It is noted that for the specified range of ω^* the thickness of the Stokes layer is less than that of the main deck and greater than that of the lower deck. However, this 'mismatch' of the regions at the trailing edge can affect only the lower-order terms in the triple-deck expansions and the fundamental problem at the trailing edge remains that of (6.6)–(6.12) with the function P replaced by its asymptotic form $P = -i\tilde{S}(M_\infty^2-1)^{\frac{1}{2}}/M_\infty$. The separation law is thus given by (6.17).

The above arguments fail when the oscillation is so rapid that $\omega^* = O(\epsilon^{-2})$, which results in the problem considered in §§ 2–5. A comparison of (4.17) and (6.6) shows that the feature distinguishing the two regimes is the appearance of the time derivative in the lower-deck equation (4.17). Clearly the critical order of magnitude of the product $h^*\omega^*$ for the occurrence of separation remains ϵ^2 but the precise law is no longer given by (6.17) and cannot be inferred from the results for a steady inclined plate. Presumably the new law specifies the minimum scaled amplitude h for separation, as a function of the scaled frequency parameter S, but a determination of the value of this function for even one value of S would require a full numerical solution of (4.17)–(4.25) and this has not been attempted.

In the incompressible case, the trailing-edge stall angle has not been determined numerically for a steady inclined plate, although Brown & Stewartson (1970) make an estimate of its value. It is expected that the arguments of this section have a direct analogue in the incompressible problem. A more detailed investigation of the flow is required, however, before this can be confirmed since the velocity components in the boundary layer must presumably be matched to the leading-order terms in the triple deck through a foredeck of streamwise length $O(\omega^{*-1})$. Further investigation is also required to consider the range $\omega^* \gg \epsilon^{-2}$; in this case the time derivative in the lower-deck equation (4.17) becomes dominant and it is possible that the triple-deck structure at the trailing edge is destroyed by the rapid oscillation.

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